1. $\alpha(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$, $\theta \in I$

$= r(\theta)(\cos\theta, \sin\theta)$

$\alpha'(\theta) = r'(\theta)(\cos\theta, \sin\theta) + r(\theta)(-\sin\theta, \cos\theta)$

$\alpha''(\theta) = (r''(\theta) - r(\theta))(\cos\theta, \sin\theta) + 2r'(\theta)(-\sin\theta, \cos\theta)$

$|\alpha'(\theta)|^2 = r'(\theta)^2 + r(\theta)^2$, $\det(\alpha', \alpha'') = 2r'(\theta)^2 - r(\theta)(r''(\theta) - r(\theta))$

$$L_a^b(\alpha) = \int_a^b |\alpha'(\theta)|\, d\theta = \int_a^b \sqrt{r'(\theta)^2 + r(\theta)^2}\, d\theta$$

$$k(\theta) = \frac{\det(\alpha', \alpha'')}{|\alpha'|^3} = \frac{2r'(\theta)^2 - r(\theta)(r''(\theta) - r(\theta))}{[r'(\theta)^2 + r(\theta)^2]^{\frac{3}{2}}}$$

$$= \frac{2r'(\theta)^2 - r(\theta)r''(\theta) + r(\theta)^2}{[r'(\theta)^2 + r(\theta)^2]^{\frac{3}{2}}}$$

2. If $|\alpha(s)| \neq 0$, $\frac{d}{ds}|\alpha(s)| = \frac{\langle \alpha'(s), \alpha(s)\rangle}{|\alpha(s)|}$

$$\frac{d^2}{ds^2}|\alpha(s)| = \frac{\langle \alpha''(s), \alpha(s)\rangle + \langle \alpha'(s), \alpha'(s)\rangle}{|\alpha(s)|} - \frac{\langle \alpha'(s), \alpha(s)\rangle}{|\alpha(s)|^2}\frac{d}{ds}|\alpha(s)|$$

Since $|\alpha(s_0)| = \max_{s\in I} |\alpha(s)|$, $\frac{\langle \alpha'(s_0), \alpha(s_0)\rangle}{|\alpha(s_0)|} = 0$

$$\frac{\langle \alpha''(s_0), \alpha(s_0)\rangle + 1}{|\alpha(s_0)|} \leq 0$$

Since $\langle \alpha'(s_0), \alpha(s_0)\rangle = 0$, $\frac{\alpha(s_0)}{|\alpha(s_0)|} = \pm N(s_0)$

then $\langle \alpha''(s_0), \pm N(s_0)\rangle + \frac{1}{|\alpha(s_0)|} \leq 0$

$\pm k(s_0) \geq \frac{1}{|\alpha(s_0)|}$

So $|k(s_0)| \geq \frac{1}{|\alpha(s_0)|}$

3. ($\Rightarrow$) Suppose α is helix

$\langle T(s), v\rangle = c_1$ for some unit vector v and constant c_1

$\frac{d}{ds}\langle T(s), v\rangle = 0$

$\langle k(s)N(s), v\rangle = 0$

Since $k(s) > 0$, $\langle N(s), v\rangle = 0$ $\forall s \in I$

$\langle -k(s)T(s) - \tau(s)B(s), v\rangle = 0$

$$v = \langle T(s), v\rangle T(s) + \cancel{\langle N(s), v\rangle}^{\,0} N(s) + \langle B(s), v\rangle B(s)$$

$$1 = |v| = \langle T(s), v\rangle^2 + \langle B(s), v\rangle^2$$

$$\langle B(s), v\rangle = c_2 \quad \text{for some constant } c_2$$

$$-k(s)\langle T(s), v\rangle - \tau(s)\langle B(s), v\rangle = 0$$

$$-k(s)c_1 - \tau(s)c_2 = 0$$

$$\tau(s) = -\frac{c_1}{c_2}k(s) \qquad (c_2 \neq 0\text{, otherwise contradiction asise})$$

($\Leftarrow$) If $\tau(s) = ck(s)$ $\forall s \in I$ for some constant c

Let $v(s) = cT(s) - B(s)$

$$v'(s) = c(k(s)N(s)) - \tau(s)N(s)$$
$$= (ck(s) - \tau(s))N(s)$$
$$= 0$$

v is constant vector

Then $\langle T(s), \frac{v(s)}{|v(s)|}\rangle = \frac{c}{\sqrt{1+c^2}}$

4. Let $\alpha(\bar{s})$ be p.h.a.l.

$\beta(s)$ be p.h.a.l.

$$\beta(s) = \alpha(s) + r(s)N(s)$$

$$T_\beta(s) = \frac{d\bar{s}}{ds}T_\alpha(s) + r'(s)N(s) + r(s)\frac{d\bar{s}}{ds}(-k(s)T_\alpha(s) - \tau(s)B_\alpha(s))$$

$$\langle T_\beta(s), N_\alpha(s)\rangle = r'(s)$$

Since $T_\beta(s) \perp N_\beta(s)$ and $N_\beta(s) \parallel N_\alpha(s)$, $r'(s) = 0$ and hence r is constant

$$T_\beta(s) = \frac{d\bar{s}}{ds}(1 - rk_\alpha(s))T_\alpha(s) - \frac{d\bar{s}}{ds}r\tau_\alpha(s)B_\alpha(s)$$

$$k_\beta(s)N_\beta(s) = \frac{d}{ds}\left(\frac{d\bar{s}}{ds}(1-rk_\alpha(s))\right)T_\alpha(s) + \left(\frac{d\bar{s}}{ds}\right)^2(1-rk_\alpha(s))k_\alpha(s)N_\alpha(s) - \frac{d}{ds}\left(\frac{d\bar{s}}{ds}r\tau_\alpha(s)\right)B_\alpha(s)$$
$$-\left(\frac{d\bar{s}}{ds}\right)^2 r\tau_\alpha(s)^2 N_\alpha(s)$$

$$0 = \langle k_\beta(s)N_\beta(s), T_\alpha(s)\rangle = \frac{d}{ds}\left(\frac{d\bar{s}}{ds}(1-rk_\alpha(s))\right)$$

$$0 = \langle k_\beta(s)N_\beta(s), B_\alpha(s)\rangle = -\frac{d}{ds}\left(\frac{d\bar{s}}{ds}r\tau_\alpha(s)\right)$$

Hence, $\frac{d\bar{s}}{ds}(1 - rk_\alpha(s)) = c_1$

$\frac{d\bar{s}}{ds}r\tau_\alpha(s) = c_2$

So $\exists$ constant A, B such that $Ak_\alpha(s) + B\tau_\alpha(s) \equiv 1$

Suppose $\exists$ constant A, B such that $A k_\alpha(s) + B \tau_\alpha(s) \equiv 1$

Let $\beta(\bar{s}) = \alpha(\bar{s}) + A N_\alpha(\bar{s})$

$$\beta'(\bar{s}) = T_\alpha(\bar{s}) + A\left(-k_\alpha(\bar{s}) T_\alpha(\bar{s}) - \tau_\alpha(\bar{s}) B_\alpha(\bar{s})\right)$$

$$= \left(1 - A k_\alpha(\bar{s})\right) T_\alpha(\bar{s}) - A \tau_\alpha(\bar{s}) B_\alpha(\bar{s})$$

$$= B \tau_\alpha(\bar{s}) T_\alpha(\bar{s}) - A \tau_\alpha(\bar{s}) B_\alpha(\bar{s})$$

Then $T_\beta(\bar{s}) = \pm\left(\frac{B}{\sqrt{A^2+B^2}} T_\alpha(\bar{s}) - \frac{A}{\sqrt{A^2+B^2}} B_\alpha(\bar{s})\right)$

$$|\beta'(\bar{s})|\, k_\beta(\bar{s}) N_\beta(\bar{s}) = \pm\left(\frac{B}{\sqrt{A^2+B^2}} k_\alpha(\bar{s}) N_\alpha(\bar{s}) - \frac{A}{\sqrt{A^2+B^2}} \tau_\alpha(\bar{s}) N_\alpha(\bar{s})\right)$$

$$= \pm\left(\frac{B}{\sqrt{A^2+B^2}} k_\alpha(\bar{s}) - \frac{A}{\sqrt{A^2+B^2}} \tau_\alpha(\bar{s})\right) N_\alpha(\bar{s})$$

Hence $N_\beta(\bar{s}) \parallel N_\alpha(\bar{s})$